Ant and honey on regular prisms and cylinder (version 4)

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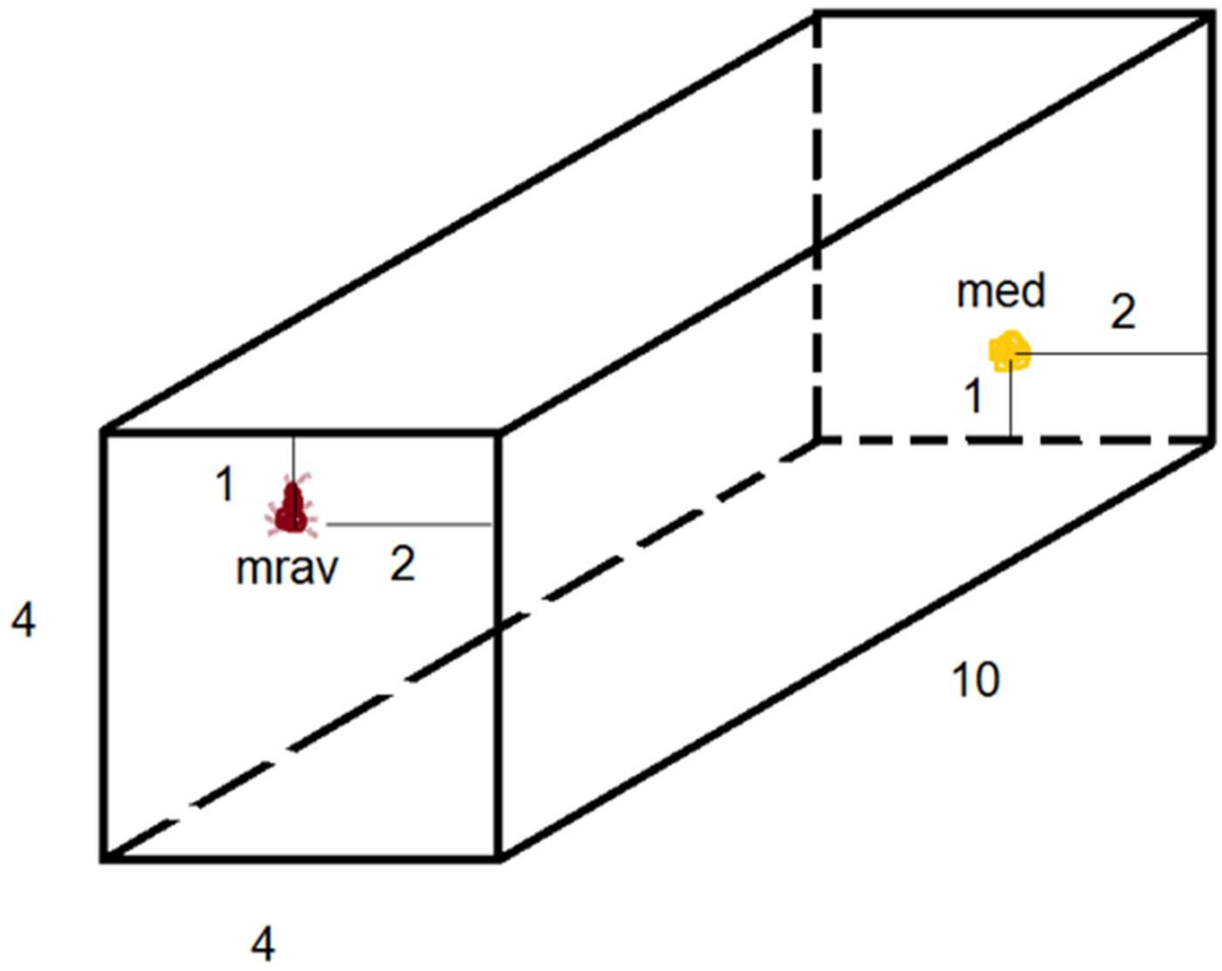
Introduction

- In 1981, a friend, Korado Korlević, gave me the task of solving the problem involving a (square) prism.
- I solved it within a few hours. However, I immediately asked myself the question: "What if it were a cylinder instead of a prism?" (with the same conditions).
- At the time, I intuitively believed there was no shorter path on the cylinder. I also thought that solving it required complex mathematics, such as calculus of variations.
- In 2013, I revisited the problem and found that a shorter path does exist on a cylinder.
- **Later, I returned to the prism problem and found that this simpler task also hides interesting "mathematical secrets."**

Ant and Honey on the 4-Prism

- Is there a shorter path (along the surface of the prism) between the ant and the drop of honey than the **"obvious path"**, which has a length of 4 + 10 = 14 (side s = 4, height h = 10)?



Ant and Honey on the 4-Prism

- The solution, i.e., a path shorter than 14, can easily be found by unfolding the prism onto a plane and rotating the top and bottom squares along the side faces in 90-degree increments.
- Out of the 4 positions of the top square and the 4 positions of the bottom square, one combination (obtained by rotating one square by 90 degrees) results in a path shorter than 14 (note: there are 4 equal paths - left or right, up or down).
- **D** represent the length of the **obvious path,**
  and **d** represent the length of the **slanted path**
  (which is shorter than the obvious path). In our case:
  d = sqrt((10 + 3)^2 + 5^2) = sqrt(194) **<** sqrt(196) = 14 = **D**
  d < D
- This is not yet a mathematical proof that this is the minimal path, but it is indeed the shortest path (it can be proven that no shorter path can pass through one or both vertices).

Ant and Honey on the 4-Prism

- **We can also observe the problem in general.**
- A square prism has a side length **s** and a height **h**.
  Thus, the obvious path between the ant and honey is:
  D = s + h
- On one base of the prism (i.e., a square), the ant is positioned (we assume the ant has no dimensions, like a point). The ant is horizontally at the center and vertically at any point (even at the edge of the base).
- On the opposite base, the drop of honey is also positioned (again, as a point). The honey is also horizontally at the center and vertically positioned so that its distance from the bottom edge is the same as the ant's distance from the top.
- **Is there a shorter path in the general case?**
- In the analysis of the problem, we will again use the Pythagorean theorem, a quadratic equation, basic trigonometry, and a bit of calculus (to find the min/max).

Ant and Honey on the 4-Prism

- **Let's first observe the case when the ant and the honey are at the edge of the prism.** We will later see that the shortest slanted path is the one where the ant starts at the edge.
- Denote with **d0** the length of the slanted path starting from the edge. For the slanted path to be shorter than the obvious path, it must hold that:

  d0 < D = (h + s)

  d0^2 < h^2 + 2 s h + s^2
- The shortest path **d0** is found when one square is rotated by 180 degrees (not 90 degrees as before), and then we have:

  d0^2 = h^2 + (2 s)^2

  d0^2 = h^2 + 4 s^2
- Thus:

  h^2 + 4 s^2 < h^2 + 2 s h + s^2

  h > 3/2 s

Ant and Honey on the 4-Prism

- **How far can the ant move from the edge while still having a slanted path shorter than the obvious path?**
- The distance of the ant from the top edge can be represented as a proportion of **s**, i.e. **p s**, where $0 \leq p < 0{,}5$.
- The distance of the honey is then **(1 - p) s**.
- **p_max** is the maximum **p** at which there is a path shorter or equal to the obvious path:

  $$p \leq p_{max}$$

- Let **k** be the ratio between h and s, i.e.:

  $$h = k\,s \text{ or } k = h / s$$

- We can calculate starting from the condition:

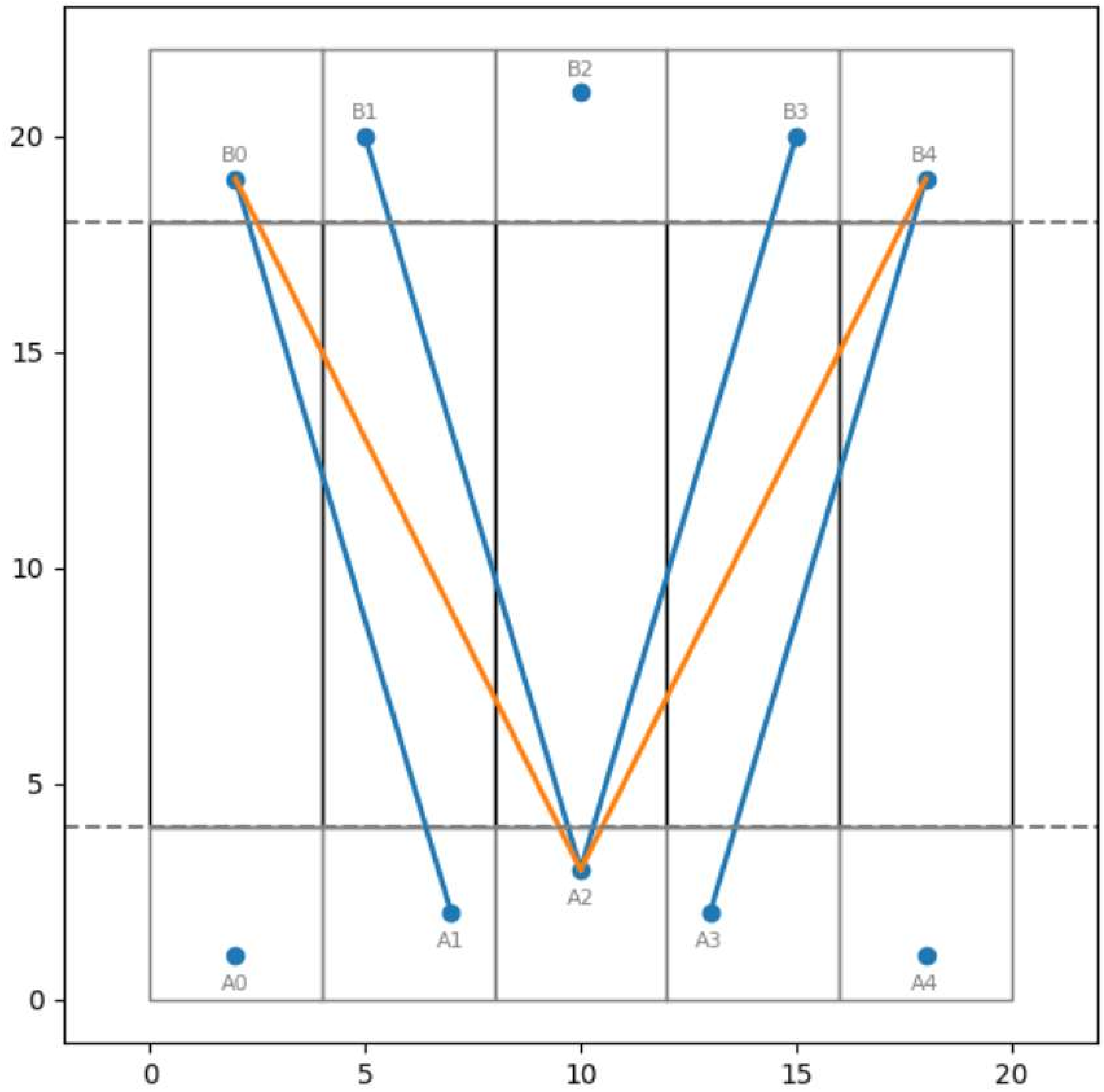
  $$d \leq D = (h + s)$$

  $$d^2 \leq (h + s)^2 = (k\,s + s)^2 = s^2 (k + 1)^2$$

- Of course, we need to find the formula for d.

Ant and Honey on the 4-Prism

- **Two shorter paths** (i.e., two classes of paths of the same length) for:
  prism height h = 14
  base height b = s = 4
  (i.e., k = 3.5)
  p = 0.25
- **Blue** (e.g. A2–B3), in the text **d1**, is shorter for the above settings
- **Orange** (A2-B4), in the text **d2**, is longer (for p = 0, it is shorter)
- **There are n+1** rectangles, but the first and last are the same.



Ant and Honey on the 4-Prism

- In the original problem, the shortest path was the one in which the square was rotated once, and in the path starting from the edge, the shortest path was obtained by rotating the square twice.
- **For 2 ≤ k ≤ 5 there are at most (depending on p) 2 shorter paths** (e.g. for k = 10 there is a third shorter path). Path d2 is shorter than path d1 when p is less than or equal to the variable **p_eq** (which depends on k), and d1 is shorter when p is greater than p_eq and less than **p_max1**.
- When **p_eq ≤ p ≤ p_max1**, the shortest length is:

d1^2 = (s/2 + s (1 - p))^2 + (s p + s k + s/2)^2

d1 = s sqrt(k^2 + k + 2 k p + 2 p^2 - 2 p + 5/2)

- When **p ≤ p_eq**, the shortest length is:

d2^2 = (s/2 + s + s/2)^2 + (s p + s k + s p)^2

d2 = s sqrt(k^2 + 4 k p + 4 p^2 + 4)

Ant and Honey on the 4-Prism

- The formula for **p_eq**, i.e. p at which the values of both minimum path formulas are equal, can be found as follows:
  for p_eq ≤ p ≤ p_max2 it holds

  d1 = s sqrt(k^2 + k + 2 k p + 2 p^2 - 2 p + 5/2)

  for p ≤ p_eq it holds

  d2 = s sqrt(k^2 + 4 k p + 4 p^2 + 4)
- By equating these formulas we get:

  k_eq = (p^2 + p + 3/4) / (1/2 - p)
- Alternatively, we get:

  2 p^2 + 2 (k + 1) p + 3/2 - k = 0

  p_eq = 1/2 (sqrt(k^2 + 4 k - 2) - k - 1)
- For example, for our case, p_eq is 0,2122…
- **For p ≤ p_eq, the shorter path is d2, and for p ≥ p_eq the shorter path is d1.**

Ant and Honey on the 4-Prism

- We obtain the formula for **p_max1** using the formula d1, i.e. the one for p_eq ≤ p ≤ p_max1:

  $$d1^2 = s^2 (k^2 + k + 2 k p + 2 p^2 - 2 p + 5/2)$$

- From the setting:

  $$d1^2 \le D^2 = s^2 (k + 1)^2$$

  is valid:

  $$2 p^2 + 2 (k - 1) p + 3/2 - k \le 0$$

  $p\_max1 = 1/2 (\sqrt{k^2 - 2} - k + 1)$

- For example, for our case (k = 3.5), p_max1 is 0.3507…

Ant and Honey on the 4-Prism

- Although formula d1 applies after $p > p_{eq}$, path d2 is still shorter than the obvious path D, up to some **p_max2**. Of course, this path is not minimal then.
- For $p \leq p_{eq}$ the following holds:

  $$d2 = s \sqrt{k^2 + 4 k p + 4 p^2 + 4}$$

- From the setting:

  $$d2^2 \leq D^2 = s^2 (k + 1)^2$$

  is valid:

  $$s^2 (k^2 + 4 k p + 4 p^2 + 4) \leq s^2 (k + 1)^2$$

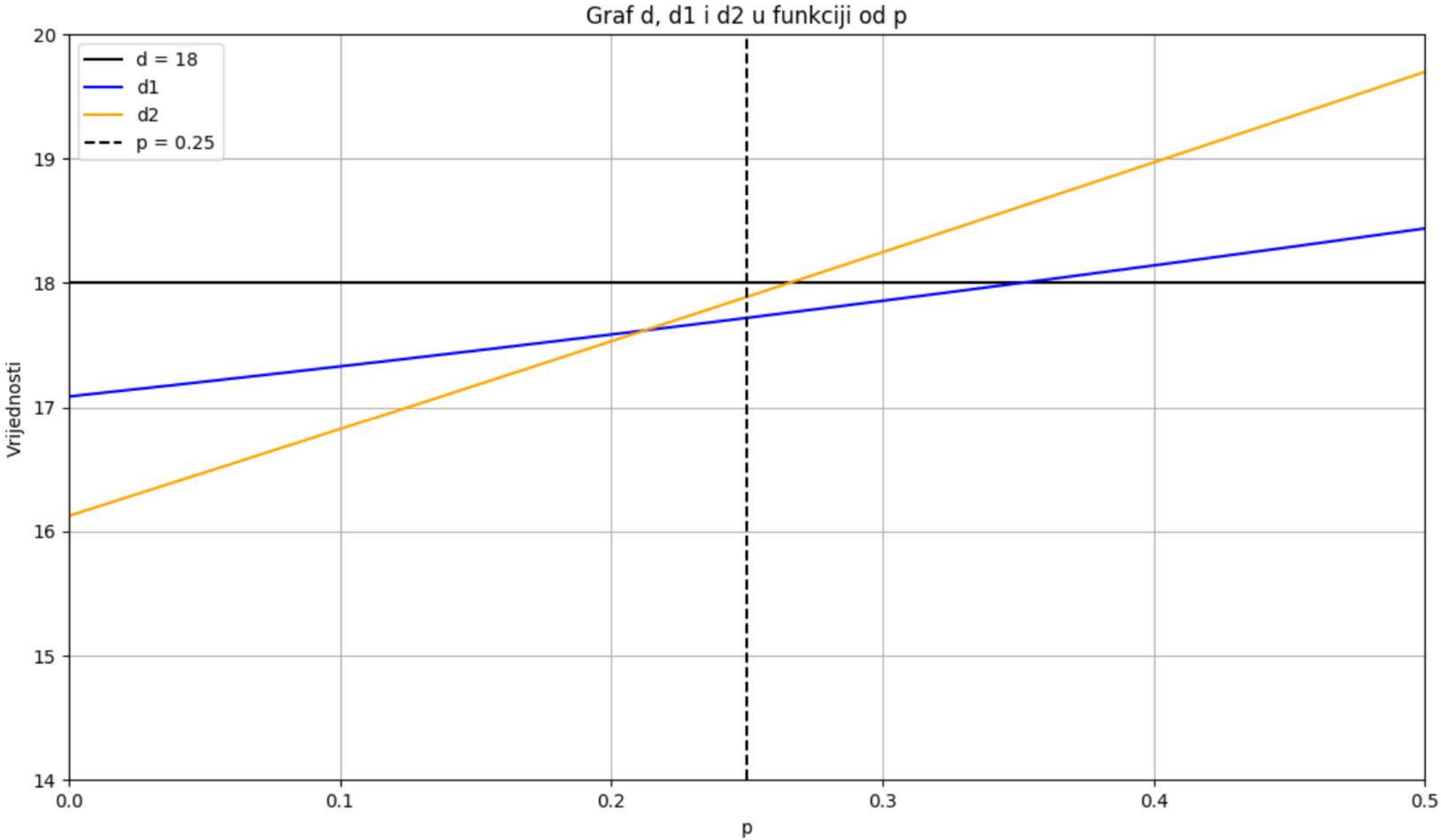
  $$2 p^2 + 2 k p + 3/2 - k \leq 0$$

  $$\mathbf{p_{max2} = 1/2 (\sqrt{k^2 + 2 k - 3} - k)}$$

- For example, for our case ($k = 3{,}5$), p_{max2} is 0,2655…

Ant and Honey on the 4-Prism

- d1 and d2, for a base of length 4 and k = 3.5



Ant and Honey on the 4-Prism

- **It is interesting to observe the relationship between the shortest path and the obvious path.**

- When $p\_eq \le p \le p\_max1$, the formula d1 is used, so:

$$d1 / D = \text{sqrt}(k^2 + k + 2 k p + 2 p^2 - 2 p + 5/2) / (k + 1)$$

- For $d1 < D$, i.e. $d1 / D < 1$, k must be:

$$k > (3/2 + 2 p^2 - 2 p) / (1 - 2 p)$$

- When $p \le p\_eq$, the formula d2 is used, so:

$$d2 / D = \text{sqrt}(k^2 + 4 k p + 4 p^2 + 4) / (k + 1)$$

- For $d2 < D$, i.e. $d2 / D < 1$, k must be:

$$k > (3/2 + 2 p^2) / (1 - 2 p)$$

Ant and Honey on the 4-Prism

- Based on the previous formulas, we will derive (two) formulas for **k_{opt}** (optimal k, for a given p), i.e. k at which the shortening is maximal.

- When $p_{eq} \le p \le p_{max2}$ it is true:

$$d1 / D = \sqrt{k^2 + k + 2 k p + 2 p^2 - 2 p + 5/2} / (k + 1)$$

- **The derivative with respect to k is:**

$$((2 p - 1) k + 4 p^2 - 6 p + 4) / (2 (k + 1) \sqrt{k^2 + (1 + 2 p) k + 2 p (p - 1) + 5/2})$$

- **Equating the numerator of the derivative to zero gives:**

$$\mathbf{k_{opt1} = (2 p^2 - 3 p + 2) / (1/2 - p)}$$

Ant and Honey on the 4-Prism

- When $p \leq p\_eq$ the following holds:

  d2 / D = sqrt(k^2 + 4 k p + 4 p^2 + 4) / (k + 1)

- **The derivative with respect to k is:**

  ((2 p - 1) k + 4 p^2 - 2 p + 4)

  / ((k + 1)^2 sqrt(k^2 + 4 p k + 4 p^2 + 4))

- **Equating the numerator of the derivative to zero gives:**

  k_opt2 = (2 p^2 - p + 2) / (1/2 - p)

Ant and Honey on the 4-Prism

- Since we have two formulas for k_opt:

  $$k\_opt1 = (2 p^2 - 3 p + 2) / (1/2 - p)$$

  $$k\_opt2 = (2 p^2 \;\; p + 2) / (1/2 - p)$$

  we ask ourselves:
  when to use the first formula, and when the second?
- Let's recall that we asked ourselves the same question with the two formulas for the minimum path. So when we equate them, we also get:

  $$\mathbf{k\_eq = (p^2 + p + 3/4) / (1/2 - p)}$$
- Let's equate this formula with the formulas for k_opt1 and k_opt2. Equating with k_opt2 does not give a solution for p (i.e., we get complex numbers). Equating with k_opt1 gives:

  $$\mathbf{p\_opt = 2 - \sqrt{11}/2} = 0{,}3416\ldots$$
- For $p \le p\_opt$ we use the formula k_opt2,
  and for $p \ge p\_opt$ we use the formula k_opt1.

Ant and Honey on the 4-Prism

- **The angle** (relative to the horizontal) at which the ant should move towards the honey drop can be expressed as a function of k and p.

- For path d1 the following holds:

tan(kut1) = (s p + s k + s/2) / (s/2 + s (1 - p))

kut1 = arctan((2 k + 2 p + 1) / (3 - 2 p))

- For path d2 the following holds:

tan(kut2) = (s p + s k + s p) / (s/2 + s + s/2)

kut2 = arctan((k + 2 p) / 2)

Ant and Honey on the 4-Prism

Recapitulation of formulas (za $2 \le k \le 5$)**:**

$D = s\,(k + 1)$

$k > 3/2$ (condition for the existence of a shorter path)

$d1 = s\,\mathrm{sqrt}(k^2 + k + 2\,k\,p + 2\,p^2 - 2\,p + 5/2)$ **$p \ge p\_eq$**

$d2 = s\,\mathrm{sqrt}(k^2 + 4\,k\,p + 4\,p^2 + 4)$ **for $p \le p\_eq$**

$p\_max1 = 1/2\,(\mathrm{sqrt}(k^2 - 2) - k + 1)$

$p\_max2 = 1/2\,(\mathrm{sqrt}(k^2 + 2\,k - 3) - k)$

$p\_eq = 1/2\,(\mathrm{sqrt}(k^2 + 4\,k - 2) - k - 1)$

$k\_opt1 = (2\,p^2 - 3\,p + 2) / (1/2 - p)$

$k\_opt2 = (2\,p^2 - p + 2) / (1/2 - p)$

$p\_opt = 2 - \mathrm{sqrt}(11)/2 = 0{,}3416\ldots$

$kut1 = \arctan((2\,k + 2\,p + 1) / (3 - 2\,p))$

$kut2 = \arctan((k + 2\,p) / 2)$

Ant and Honey on the 3-Prism

- So far we have looked at a prism whose base is a square (abbreviated: 4-prism). It is interesting to analyze a prism whose base is an equilateral triangle (3-prism).
- In a 3-prism, the apparent path has a length that is the sum of the height of the prism (h) and the height of the base (b), which here is equal to the height of the triangle (v).
- If we represent the height of the 3-prism as (the side of the triangle is s):

  h = k b = k v = k s sqrt(3)/2

  we get the following:

  D = h + b = k s sqrt(3)/2 + s sqrt(3)/2

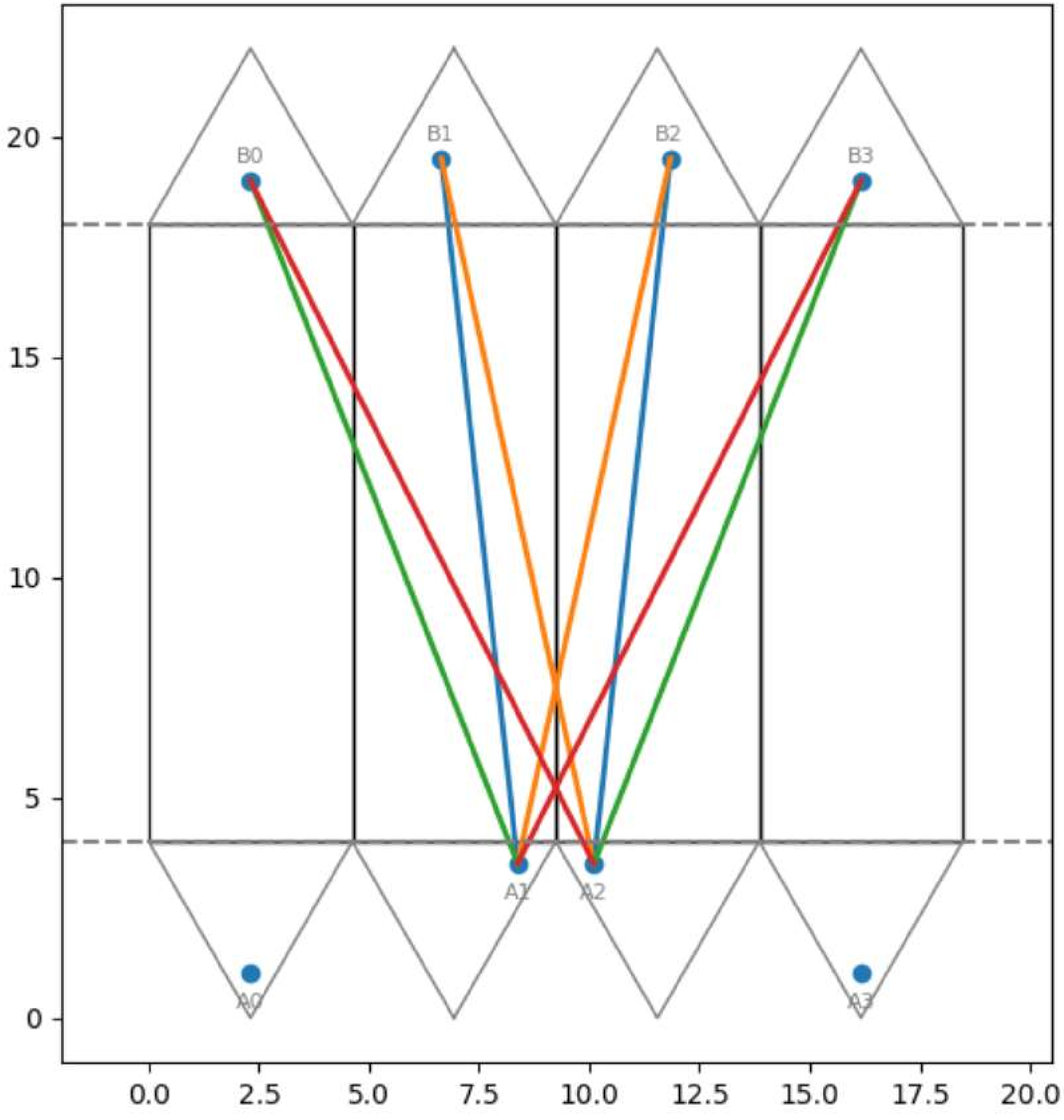
  D = s sqrt(3)/2 (k + 1) = b (k + 1)

- **For odd polygons**, the general formula for b (the bisector from the vertex to the midpoint of the opposite side) is:

  b = s/2 cot(π/(2n))

Ant and Honey on the 3-Prism

- **Four shorter paths** (that is, four classes) for settings:
  prism height h = 14
  base height b = 4
  p = 0.25
- **Blue** is the shortest, in text **d1**
- **Orange** is longer, in text **d2**
- **Green** is even longer, in text **d3**
- **Red** is the longest, in text **d4**



Ant and Honey on the 3-Prism

- **There are at most (depending on p) 4 paths shorter than the obvious path D (for $2 \leq k \leq 5$).**

- Length of path d1:

d1^2 = (s (1/2 - p) 3/2)^2

+ (s p sqrt(3)/4 + s k sqrt(3)/2 + s (1 - p) sqrt(3)/4)^2

d1 = s sqrt(3)/2 sqrt(k^2 + k + 3 p^2 - 3 p + 1)

or (since for a 3-prism b = s sqrt(3)/2)

d1 = b sqrt(k^2 + k + 3 p^2 - 3 p + 1)

- **The path d1 is always shorter than D, for every $k > 0$ and $p \geq 0$.**

Ant and Honey on the 3-Prism

- Length of path d2:

d2^2 = (s p 3/4 + s (1 - p) 3/4)^2

+ (s p sqrt(3)/4 + s k sqrt(3)/2 + s (1 - p) sqrt(3)/4)^2

d2 = b sqrt(k^2 + k + 1)

- **The path d2 does not depend on p!**

- Path d2 is always shorter than D, for every k > 0 and p ≥ 0.

- Path d2 is
  equal to path d1 for p = 0,
  and longer than path d1 for p > 0.

Ant and Honey on the 3-Prism

- Length of path d3:

$d3^2 = (s\ 3/2 - s\ p\ 3/4)^2$

$+ (s\ p\ \sqrt{3}/2 + s\ k\ \sqrt{3}/2 + s\ p\ \sqrt{3}/4)^2$

$d3 = b\ \sqrt{k^2 + 3\ k\ p + 3\ p^2 - 3\ p + 3}$

- d3 is shorter than the obvious path D when:

$k > (3\ p^2 - 3\ p + 2) / (2 - 3\ p)$

- The formula for **p_{max3}** is:

$p_{max3} = 1/6\ (\sqrt{9\ k^2 + 6\ k - 15} - 3\ k + 3)$

Ant and Honey on the 3-Prism

- Length of path d4:

$d4^2 = (s\,3/2 + s\,p\,3/4)^2$

$\quad + (s\,p\,\sqrt{3}/2 + s\,k\,\sqrt{3}/2 + s\,p\,\sqrt{3}/4)^2$

$d4 = b\,\sqrt{k^2 + 3\,k\,p + 3\,p^2 + 3\,p + 3}$

- d4 is shorter than the obvious path D when:

$k > (3\,p^2 + 3\,p + 2) / (2 - 3\,p)$

- The formula for **p_max4** is:

$p_{max4} = 1/6\,(\sqrt{9\,k^2 + 42\,k - 15} - 3\,k - 3)$

Ant and Honey on the 3-Prism

- For which p_eq paths are of equal length?
- For d1 to be longer than d3, the following holds:

  $p < 1/3\ (1 - 2 / k)$ tj. **$p\_eq1\_3 = 1/3\ (1 - 2 / k)$**

  so this condition also holds: $1 - 2 / k > 0 \Rightarrow$ **$k > 2$**
- Equivalently: **$k\_eq1\_3 = 2 / (1 - 3\ p)$**
- If $k > 2$ AND $p < 1/3\ (1 - 2 / k)$ the shorter is d3, otherwise d1.
- For d1 and d4 the following holds:

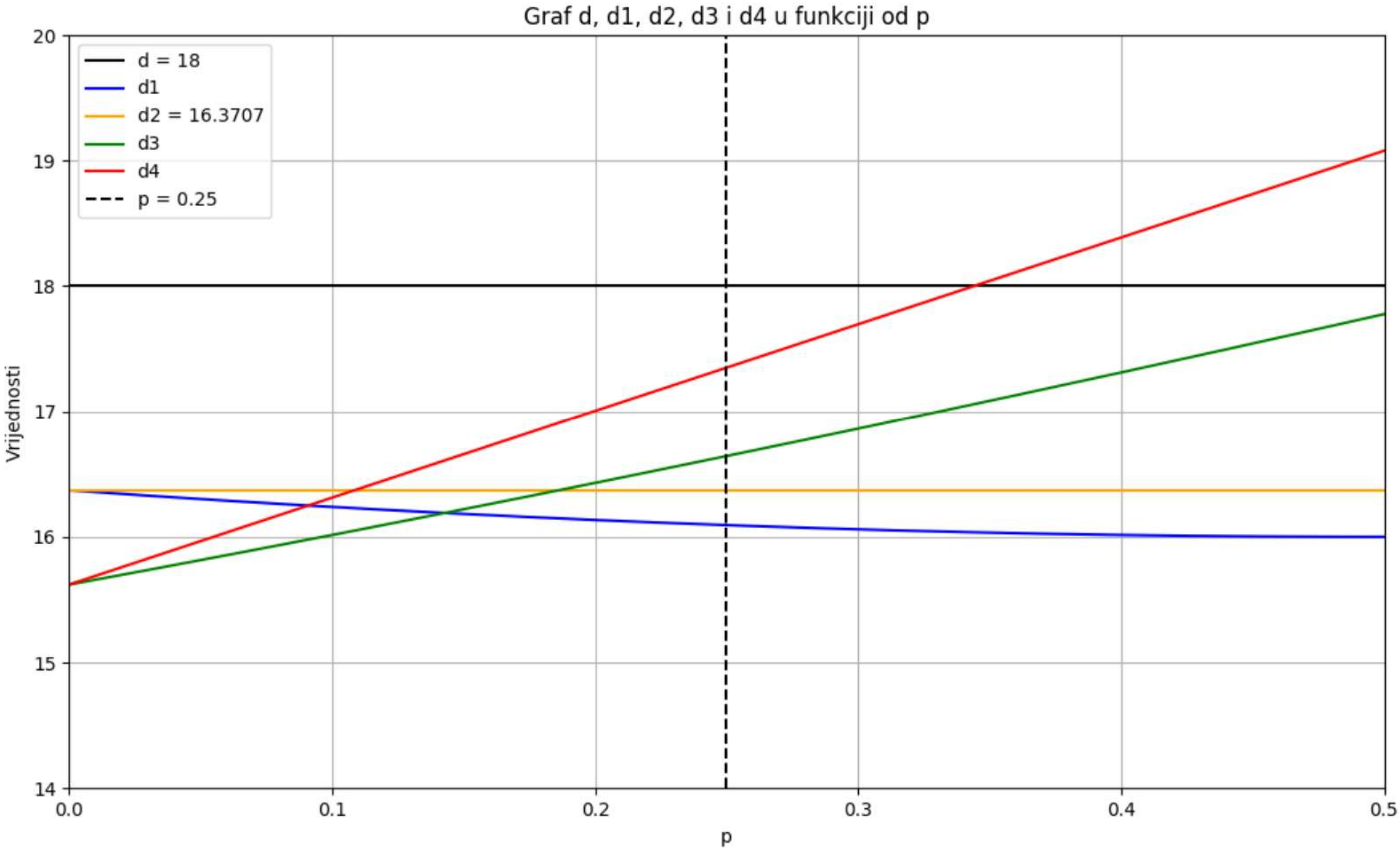
  $p\_eq1\_4 = (k - 2) / (3\ k + 6)$
- For d2 and d3 the following holds:

  $p\_eq2\_3 = 1/6\ (\mathrm{sqrt}(9\ k^2 - 6\ k - 15) - 3\ k + 3)$
- For d2 and d4 the following holds:

  $p\_eq2\_4 = 1/6\ (\mathrm{sqrt}(9\ k^2 + 30\ k - 15) - 3\ k - 3)$
- d3 is always shorter than d4, except for $p = 0$,
  when they are equal

Ant and Honey on the 3-Prism

- d1, d2, d3 and d4, or the base b = 4 and k = 3,5



Ant and Honey on the 3-Prism

- **The angle** (relative to the horizontal) at which the ant should move towards the honey drop, for paths d1, d2:

tan(alfa1) = (s p sqrt(3)/4 + s k sqrt(3)/2 + s (1 - p) sqrt(3)/4)
/ (s (1/2 - p) 3/2)

kut1 = alfa1 - π/3

kut1 = arctan(sqrt(3)/3 (2 k + 1) / (1 - 2 p)) - π/3

tan(alfa2) = (s p sqrt(3)/4 + s k sqrt(3)/2 + s (1 - p) sqrt(3)/4)
/ (s p 3/4 + s (1 - p) 3/4)

kut2 = 2/3 π - alfa2

kut2 = 2/3 π - arctan(sqrt(3)/3 (2 k + 1))

Ant and Honey on the 3-Prism

- **The angle** (relative to the horizontal) at which the ant should move towards the honey drop, for paths d3, d4:

tan(alfa3) = (s p sqrt(3)/2 + s k sqrt(3)/2 + s p sqrt(3)/4)

/ (s 3/2 - s p 3/4)

kut3 = alfa3 - π/3

kut3 = arctan(sqrt(3)/3 (2 k + 3 p) / (2 - p)) - π/3

tan(alfa4) = (s p sqrt(3)/2 + s k sqrt(3)/2 + s p sqrt(3)/4)

/ (s 3/2 + s p 3/4) /

kut4 = 2/3 π - alfa4

kut4 = 2/3 π - arctan(sqrt(3)/3 (6 k + 9 p) / (6 + 3 p))

Ant and Honey on the 3-Prism

Formula recapitulation (za $2 \leq k \leq 5$):

D = b (k + 1)

d1 = b sqrt(k^2 + k + 3 p^2 - 3 p + 1)

d2 = b sqrt(k^2 + k + 1) **(does not depend on p)**

d3 = b sqrt(k^2 + 3 k p + 3 p^2 - 3 p + 3)

d4 = b sqrt(k^2 + 3 k p + 3 p^2 + 3 p + 3)

p_max1 – there is no limit, d1 is always shorter than D

p_max2 – there is no limit, d2 does not depend on p

p_max3 = 1/6 (sqrt(9 k^2 + 6 k - 15) - 3 k + 3)

p_max4 = 1/6 (sqrt(9 k^2 + 42 k - 15) - 3 k - 3)

Ant and Honey on the 3-Prism

Formula recapitulation (za $2 \leq k \leq 5$):

p_eq1_2 = 0, and for $p > 0$ the path d1 is shorter than d2

p_eq1_3 = 1/3 (1 - 2 / k) (must be $k > 2$)

p_eq1_4 = (k - 2) / (3 k + 6)

p_eq2_3 = 1/6 (sqrt(9 k^2 - 6 k - 15) - 3 k + 3)

p_eq2_4 = 1/6 (sqrt(9 k^2 + 30 k - 15) - 3 k - 3)

p_eq3_4 = 0, and for $p > 0$ the path d1 is shorter than d4

kut1 = arctan(sqrt(3)/3 (2 k + 1) / (1 - 2 p)) - π/3

kut2 = 2/3 π - arctan(sqrt(3)/3 (2 k + 1))

kut3 = arctan(sqrt(3)/3 (2 k + 3 p) / (2 - p)) - π/3

kut4 = 2/3 π - arctan(sqrt(3)/3 (6 k + 9 p) / (6 + 3 p))

Ant and Honey on the 6-Prism

- Now we analyze a prism whose base is a regular hexagon (6-prism). With a 6-prism, the apparent path has a length that is the sum of the height of the prism (h) and the two heights of the triangle (v).
- If we show the height of the 6-prism as (b is the length of the base, v is the height of the triangle):

  $h = k\ b = k\ (2\ v)$

  where $v = s\ \mathrm{sqrt}(3)/2$, we get the following:

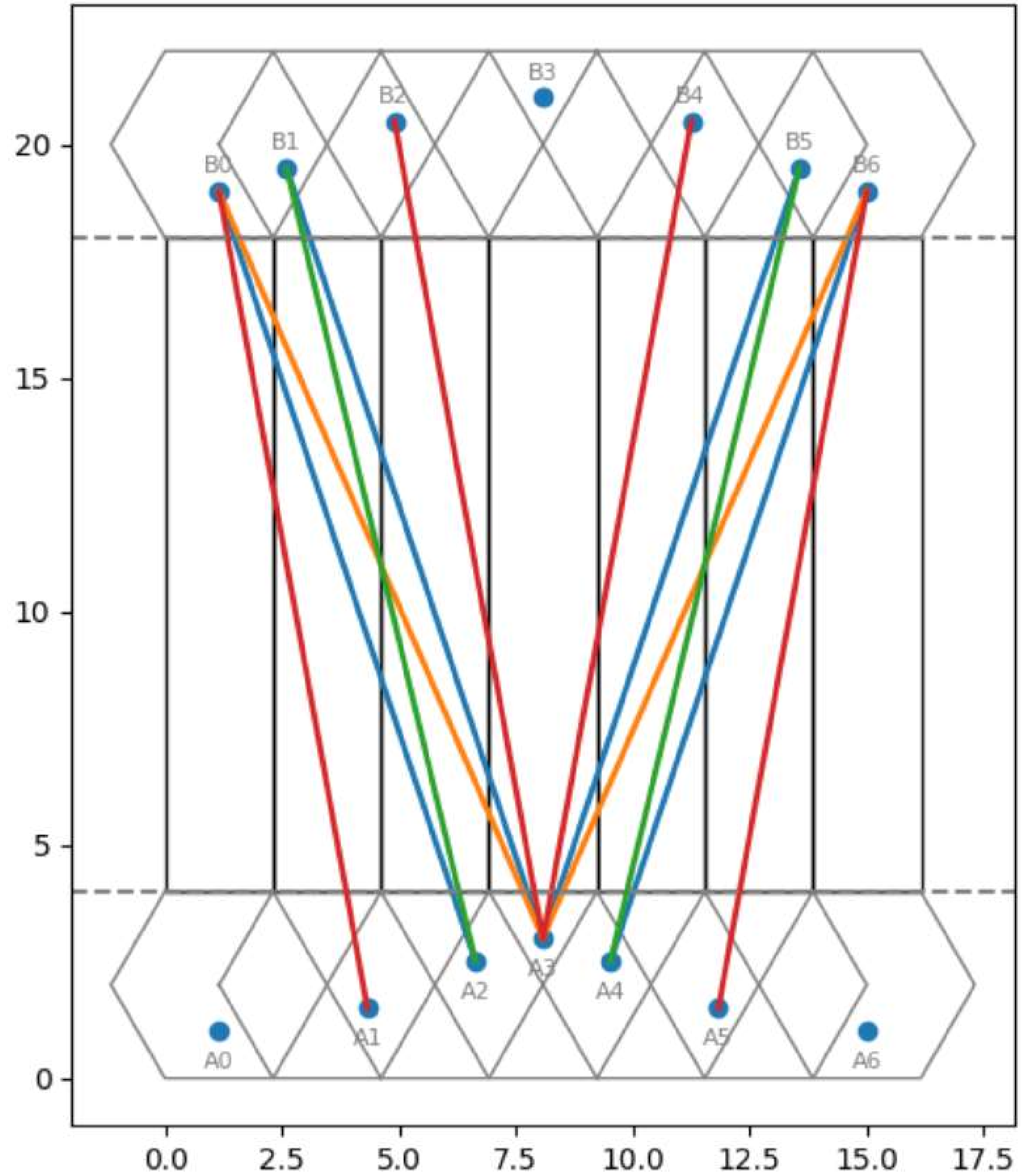
  $D = h + b = b\ (k + 1) = 2\ v\ (k + 1) = s\ \mathrm{sqrt}(3)\ (k + 1)$

  $D = b\ (k + 1)$
- **For even polygons**, the general formula for b (the bisector from the midpoint of one side to the midpoint of the opposite side) is:

  $b = s \cot(\pi/n)$

Ant and Honey on the 6-Prism

- **Four shorter paths** (that is, four classes) for settings:
  prism height h = 14
  base height b = 2 v = 4
  p = 0.25
- **Blue** is the shortest, in text **d1**
- **Orange** is longer, in text **d2**
- **Green** is even longer, in text **d3**
- **Red** is the longest, in text **d4**



Ant and Honey on the 6-Prism

- **There are at most (depending on p) 4 paths shorter than the obvious path D (for $2 \leq k \leq 5$).**

- Length of path d1:

d1^2 = (2 s + 3/4 (1 - 2 p) s)^2

+ (s sqrt(3) p + s sqrt(3) k + s sqrt(3) (p / 2 + 1/4))^2

d1 = b sqrt(k^2 + 1/2 k + 3 k p + 3 p^2 - 2 p + 31/12)

- d1 is shorter than the obvious path D when:

k > (2 p^2 - 4/3 p + 19/18) / (1 - 2 p)

- The formula for **p_max1** is:

p_max1 = 1/6 (sqrt(9 k^2 + 6 k - 15) - 3 k + 2)

Ant and Honey on the 6-Prism

- Length of path d2:

$d2^2 = (3\ s)^2 + (2\ \sqrt{3}\ p\ s + k\ \sqrt{3}\ s)^2$

$d2 = b\ \sqrt{k^2 + 4\ k\ p + 4\ p^2 + 3}$

- d2 is shorter than the obvious path D when:

$k > (2\ p^2 + 1) / (1 - 2\ p)$

- The formula for **p_max2** is:

$p_{max2} = 1/2\ (\sqrt{k^2 + 2\ k - 2} - k)$

Ant and Honey on the 6-Prism

- Length of path d3:

$$d3^2 = (s + 2\ 3/4\ (1 - 2\ p)\ s)^2$$
$$+ (2\ \sqrt{3}\ (p / 2 + 1/4)\ s + k\ \sqrt{3}\ s)^2$$

$$\mathbf{d3 = b\ \sqrt{k^2 + k + 2\ k\ p + 4\ p^2 - 4\ p + 7/3}}$$

- d3 is shorter than the obvious path D when:

$$\mathbf{k > 4/3\ (3\ p^2 - 3\ p + 1) / (1 - 2\ p)}$$

- The formula for **p_max3** is:

$$\mathbf{p_{max3} = 1/12\ (\sqrt{9\ k^2 - 12} - 3\ k + 6)}$$

Ant and Honey on the 6-Prism

- Length of path d4:

$d4^2 = (s + 3/4 (1 - 2 p) s)^2$

$+ (p \sqrt{3} s + k \sqrt{3} s + \sqrt{3}/2 s + \sqrt{3}/4 (1 - 2 p) s)^2$

$d4 = b \sqrt{k^2 + 3/2 k + k p + p^2 - p + 19/12}$

- d4 is shorter than the obvious path D when:

$k > (2 p^2 - 2 p + 7/6) / (1 - 2 p)$

- The formula for **p_max4** is:

$p_{max4} = 1/6 (\sqrt{9 k^2 - 12} - 3 k + 3)$

Ant and Honey on the 6-Prism

- For which **p_eq** paths are of equal length?

$$p\_eq1\_2 = 1/6\,(\mathrm{sqrt}(9\,k^2 + 54\,k + 21) - 3\,k - 6)$$

$$p\_eq1\_3 = 1/2\,(-\mathrm{sqrt}(k^2 + 2\,k + 5) + k + 2)$$

$$p\_eq1\_4 = 1/4\,(\mathrm{sqrt}(4\,k^2 + 4\,k - 7) - 2\,k + 1)$$

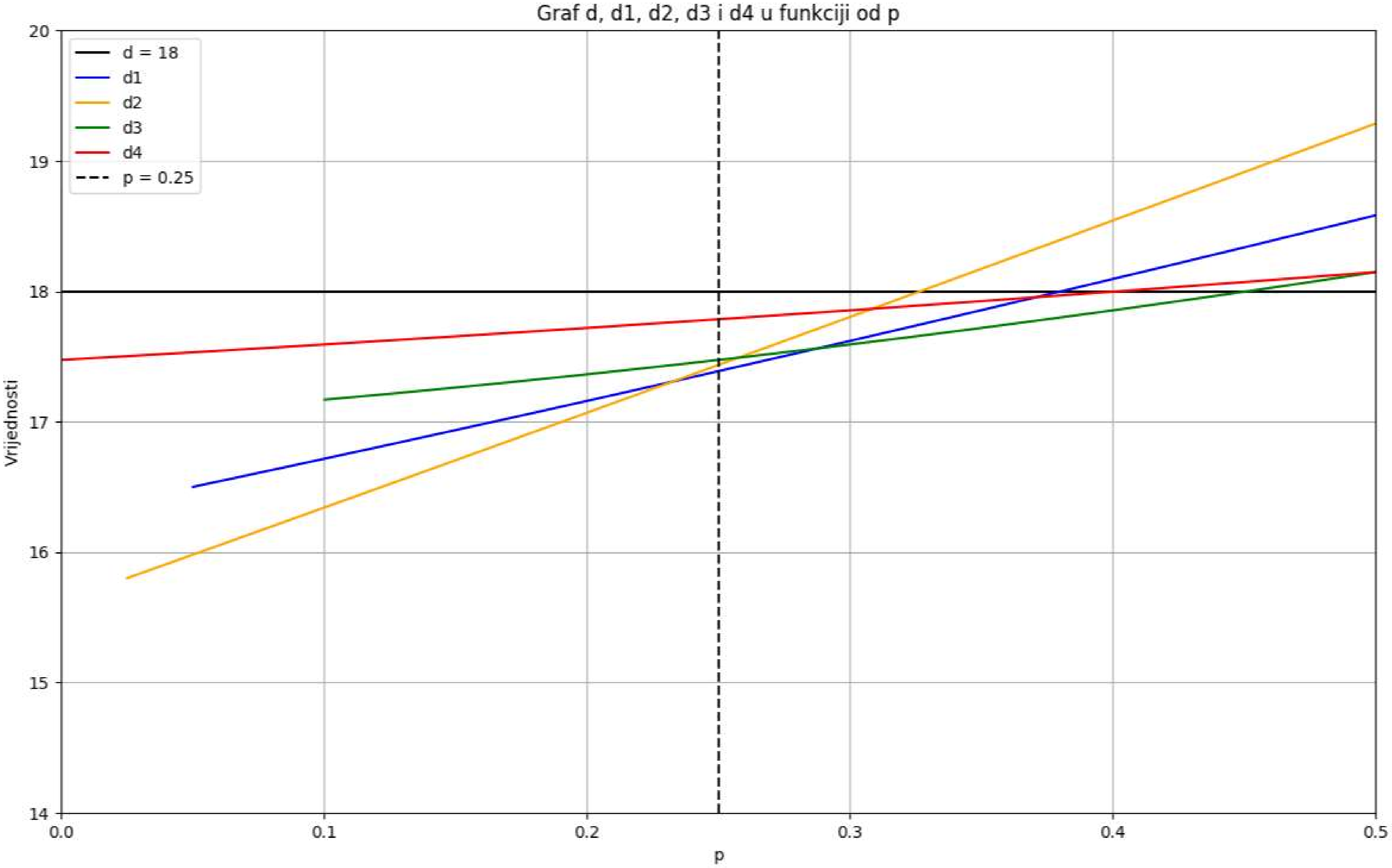
$$p\_eq2\_3 = (3\,k - 2) / (6\,k + 12)$$

$$p\_eq2\_4 = 1/6\,(\mathrm{sqrt}(9\,k^2 + 24\,k - 16) - 3\,k - 1)$$

$p\_eq3\_4 = 1/2 - k / 3$ if $k < 3/2$, otherwise $1/2$

Ant and Honey on the 6-Prism

- d1, d2, d3 and d4, for the base b = 4 and k = 3,5



Ant and Honey on the 6-Prism

- **The angle** (relative to the horizontal) at which the ant should move towards the honey drop, for paths d1, d2:

tan(kut1) = (s sqrt(3) p + s sqrt(3) k + s sqrt(3) (p / 2 + 1/4))
/ (2 s + 3/4 (1 - 2 p) s)

kut1 = arctan(sqrt(3) (4 k + 6 p + 1) / (11 - 6 p))

tan(kut2) = (2 sqrt(3) p s + k sqrt(3) s)
/ (3 s)

kut2 = arctan(sqrt(3)/3 (k + 2 p))

Ant and Honey on the 6-Prism

- **The angle** (relative to the horizontal) at which the ant should move towards the honey drop, for paths d3, d4:

tan(alfa3) = (2 sqrt(3) (p / 2 + 1/4) s + k sqrt(3) s)
/ (s + 2 3/4 (1 - 2 p) s)

kut3 = alfa3 - π/3

kut3 = arctan(sqrt(3) (2 k + 2 p + 1) / (5 - 6 p)) - π/3

tan(kut4) = k sqrt(3) s + sqrt(3)/2 s + sqrt(3)/4 (1 - 2 p) s)
/ (s + 3/4 (1 - 2 p) s) / (p sqrt(3) s)

kut4 = arctan(sqrt(3) (4 k + 2 p + 3) / (7 - 6 p))

Ant and Honey on the 6-Prism

- **Formula recapitulation** (za $2 \leq k \leq 5$):

$$D = b (k + 1)$$

$$d1 = b \operatorname{sqrt}(k^2 + 1/2\, k + 3\, k\, p + 3\, p^2 - 2\, p + 31/12)$$

$$d2 = b \operatorname{sqrt}(k^2 + 4\, k\, p + 4\, p^2 + 3)$$

$$d3 = b \operatorname{sqrt}(k^2 + k + 2\, k\, p + 4\, p^2 - 4\, p + 7/3)$$

$$d4 = b \operatorname{sqrt}(k^2 + 3/2\, k + k\, p + p^2 - p + 19/12)$$

$$p\_max1 = 1/6\, (\operatorname{sqrt}(9\, k^2 + 6\, k - 15) - 3\, k + 2)$$

$$p\_max2 = 1/2\, (\operatorname{sqrt}(k^2 + 2\, k - 2) - k)$$

$$p\_max3 = 1/12\, (\operatorname{sqrt}(9\, k^2 - 12) - 3\, k + 6)$$

$$p\_max4 = 1/6\, (\operatorname{sqrt}(9\, k^2 - 12) - 3\, k + 3)$$

Ant and Honey on the 6-Prism

- **Formula recapitulation** (za $2 \le k \le 5$):

p_eq1_2 = 1/6 (sqrt(9 k^2 + 54 k + 21) - 3 k - 6)

p_eq1_3 = 1/2 (- sqrt(k^2 + 2 k + 5) + k + 2)

p_eq1_4 = 1/4 (sqrt(4 k^2 + 4 k - 7) - 2 k + 1)

p_eq2_3 = (3 k - 2) / (6 k + 12)

p_eq2_4 = 1/6 (sqrt(9 k^2 + 24 k - 16) - 3 k - 1)

p_eq3_4 = 1/2 - k / 3 if k < 3/2, otherwise ½

kut1 = arctan(sqrt(3) (4 k + 6 p + 1) / (11 - 6 p))

kut2 = arctan(sqrt(3)/3 (k + 2 p))

kut3 = arctan(sqrt(3) (2 k + 2 p + 1) / (5 - 6 p)) - π/3

kut4 = arctan(sqrt(3) (4 k + 2 p + 3) / (7 - 6 p))

Ant and Honey on the 5-Prism

- Now we analyze a prism whose base is a regular pentagon (5-prism).
- In a 5-prism, the apparent path has a length that is the sum of the height of the prism (h) and the height of the base (b):

$$D = h + b = b\,k + b = b\,(k + 1)$$

- The length of the base b is the sum of the height of the triangle (v) and the side of the triangle (t):

$$b = v + t$$

$$b = s/2 \tan(3/10\ \pi) + s/2 / \cos(3/10\ \pi)$$

- We know that for odd polygons in general

$$b = s/2 \cot(\pi/(2n))$$

so specifically (n = 5) we can also write it like this

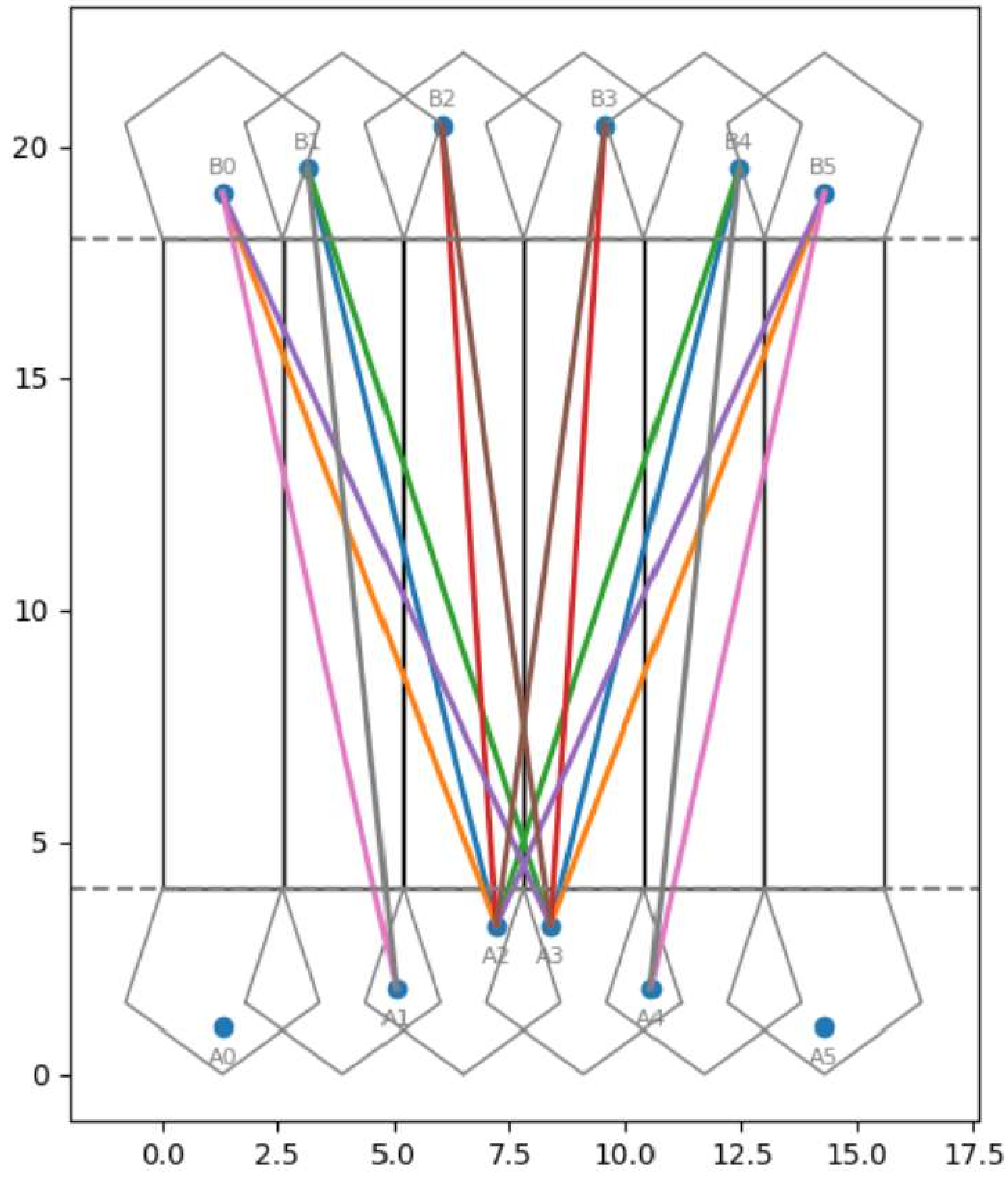
$$b = s/2 \cot(\pi/10)$$

- So:

$$\mathbf{D = b\,(k + 1) = s/2 \cot(\pi/10)\,(k + 1)}$$

Ant and Honey on the 5-Prism

- **Eight shortest paths (or, eight classes)**
  for the settings:
  prism height h = 14
  base height b = 4
  p = 0.25
- Blue is the shortest, then orange, green, red, purple, brown, pink, gray
- As with the triangular prism, there is a path that does not depend on the distance of points A and B from the edge
  - here it is brown.



Ant and Honey on the 5-Prism

- There are at most 8 paths shorter than the obvious path D (for $2 \le k \le 5$), but we will show formulas only for d1 (the shortest path for the given parameters) and d6 (the path independent of p).

d1 = b sqrt(
((10 - 3 sqrt(5)) / sqrt(10 - 2 sqrt(5)) - ((3 + sqrt(5))/8) sqrt(10 - 2 sqrt(5)) p)^2
+ (k + (sqrt(5) - 1)/4 + (sqrt(5)/2) p)^2
)

d6 = b sqrt((cos(3/10 π))^2 +(k + sin(3/10 π))^2)
d6 = b sqrt(k^2 + (1 + sqrt(5))/2 k + 1)

- Angle (relative to the horizontal), for paths d1, d6:

kut1 = arctan(
(k + (sqrt(5) - 1)/4 + (sqrt(5)/2) p) /
((10 - 3 sqrt(5)) / sqrt(10 - 2 sqrt(5)) - ((3 + sqrt(5))/8) sqrt(10 - 2 sqrt(5)) p)
) – π/5

kut6 = 4/5 π - arctan((k + sin(3/10 π)) / cos(3/10 π))

Ant and Honey on the Cylinder

- The basic idea is that the upper circle (honey) rolls on the cylinder shell, and accordingly the lower circle (ant) also rolls.
- **Then, these 4 points must be on the same line:**
  - honey and ant (imagined as points)
  - points of contact of the upper/lower circles with the mantle.
- If r = 2 and v = 10 and the ant is located at 1 from the upper edge, in order for at least three points (both touch points and the ant or honey) to be on the same line, the angle (x1) of the rolling of the lower circle depends on the angle (x) of rolling the upper circle like this:
  $x1 = x - 5 \sin(x) / (\cos(x) + 2)$
- For all 4 points to be on the line, it must also be:
  $x1 = \pi - x$
- Then x can be calculated using a numerical method from:
  $2x - 5 \sin(x) / (\cos(x) + 2) - \pi = 0$

Ant and Honey on the Cylinder

- The obvious path D has a length that is the sum of the height of the prism and 2 radii:
  $D = h + 2r = 10 + 4 = 14$

- If $r = 2$, $h = 10$ and the ant is 1 from the top edge, the formula for the smallest distance between the ant and the honey is (the roll angle of the top circle, x, was previously calculated):

d = 2 sqrt(
(sin(x) + 5 sin(x) / (cos(x) + 2))^2
+
(cos(x) + 7)^2
)

- The minimum length ≈ 13,32
  for the roll angle of the upper circle ≈ 2,64 radians.

Ant and Honey on the Cylinder

- The formulas for the cylinder can also be generalized.

- Any radius (r) can be taken, and the height of the roller (h) can be shown as k * height of the base
(diameter of the circle, 2 * r):
h = k * 2 * r

- Also, the initial position of the ant can be on the segment **p 2 r**, i.e. from the edge (p = 0) to the center (p = 0,5).

- Of course, there is no path from the center that is shorter than the obvious one, because all paths from the center are obvious paths.

Ant and Honey on the Cylinder

- The obvious path has a length that is the sum of the height of the prism and 2 radii:
  $D = h + 2r = k\,2r + 2r$
  $D = 2r(k + 1)$
- As with the prism, the shortest path is the one starting from the edge (ie when $p = 0$):
  $d0 = r\,\mathrm{sqrt}(\pi^2 + 4 * k^2)$
- When $p = 0$, the condition for the existence of a shorter path is obtained from the expression **$d0 < D$:**
  $k > \pi^2 / 8 - 1/2$ $(\approx 0{,}73)$
- When $p = 0$, the **optimal k** is obtained (as with prisms) by searching the **minimum for d0 / D:**
  $k_{opt} = \pi^2 / 4$ $(\approx 2{,}47)$
  and then $d0 / D \approx 0{,}84$

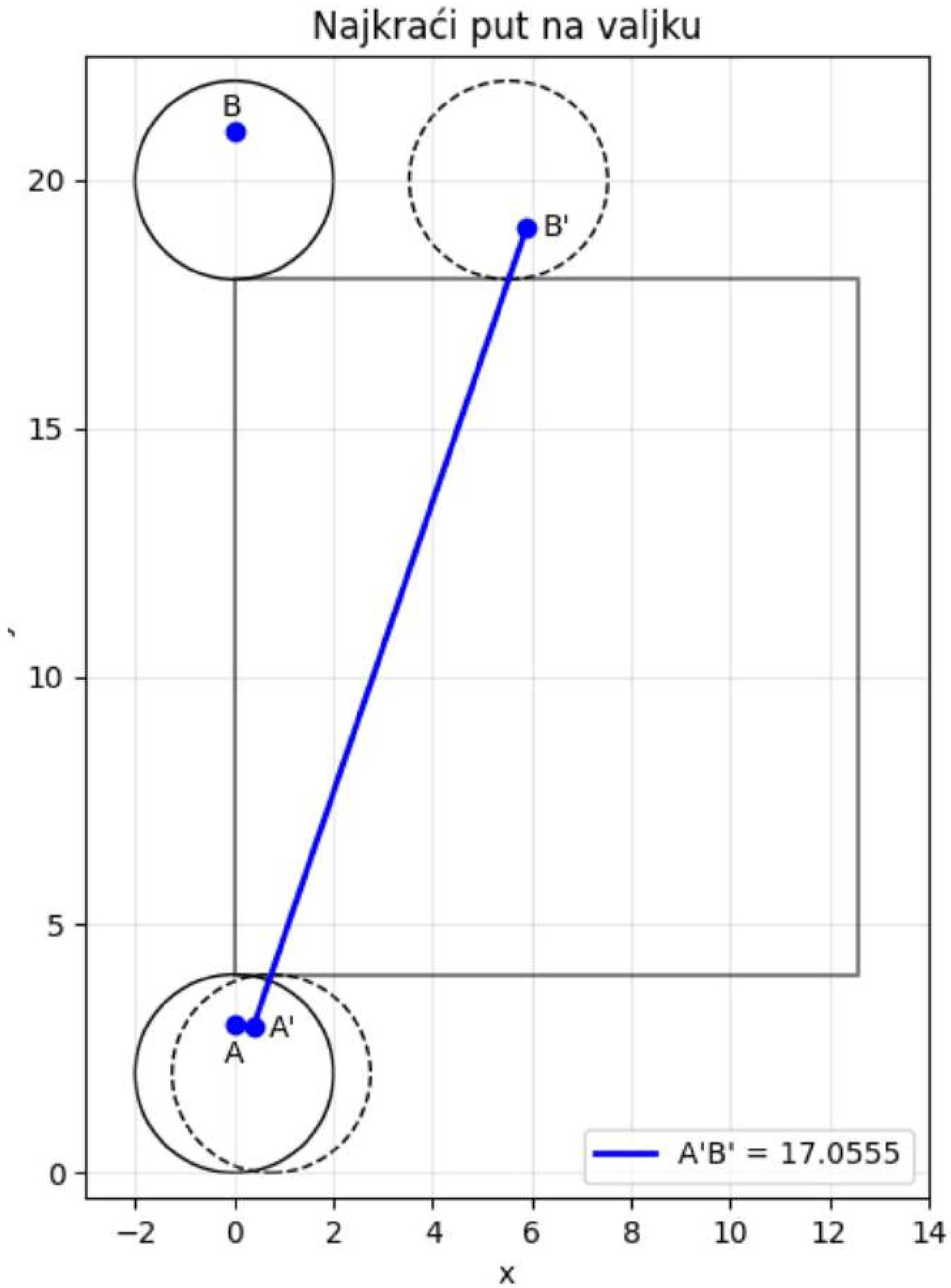
Ant and Honey on the Cylinder

- In order for at least three points (both touch points and the ant or honey) to be on the same line, the angle (x1) of the rolling of the lower circle depends on the angle (x) of rolling the upper circle like this:
  x1 = x - 2 k sin(x) / (cos(x) + 1 / (1 - 2 p))
- For all 4 points to be on the line, it must also be:
  x1 = π - x
- Then x can be calculated using a numerical method from:
  2 x - 2 k sin(x) / (cos(x) + 1 / (1 - 2 p)) - π = 0
- For a general value of r, where k > (π^2/8 -1/2) and 0 < p < 1/2, the formula for the smallest distance is
  d = 2 r sqrt(
  ((1 - 2 p) sin(x) + k sin(x) / (cos(x) + 1 / (1 – 2 p)))^2
  + ((1 - 2 p) cos(x) + k + 1)^2
)

Ant and Honey on the Cylinder

- For the settings:
  cylinder height h = 14
  base height b = 2
  r = 4 (i.e. k = 3.5)
  p = 0.25

- It is obtained:
  x ≈ 2,77 rad
  x1≈ 0,37 rad
  d ≈ 17,06



Ant and Honey on the Cylinder

- Below are the formulas for the angle (relative to the horizontal) at which the ant should move towards the drop of honey.
- For **p = 0** it holds:
  tan (alfa) = k 2 r / (r π) = 2 k / π
  angle = π/2 - alfa
  angle = π/2 - arctan (2 k / π)

- For **p > 0** it holds:
  (x and x1 are the rotation angles of the upper and lower circle):
  angle = arctan ((cos(x1) - 1 + 2 p) / sin(x1))
  so (since for the minimum path should hold: x1 = π - x):
  angle = arctan ((cos(π - x) - 1 + 2 p) / sin(π - x))
  angle = arctan ((-cos(x) - 1 + 2 p) / sin(x))

Ant and Honey on the Cylinder

- Numerical analysis shows that
  there is a minimal k also when p > 0.
- We require that **d < D**, i.e.:

2 r sqrt(
 ((1 - 2 p) sin(x) + k sin(x) / (cos(x) + 1 / (1 - 2 p)))^2
 + ((1 - 2 p) cos(x) + k + 1)^2
)
< (2 r (k + 1))

- From
  2 x - 2 k sin(x) / (cos(x) + 1 / (1 - 2 p)) - π = 0
  we get
  k = (x - π/2) (cos(x) + 1 / (1 - 2 p)) / sin(x)
  and include that in the above inequality.

Ant and Honey on the Cylinder

- **Therefore, we require that:**

sqrt(

((1 - 2 p) sin(x) + (x - π/2))^2

\+

((1 - 2 p) cos(x) + (x - π/2)

(cos(x) + 1 / (1 - 2 p)) / sin(x) + 1)^2

)

< ((x - π/2) (cos(x) + 1 / (1 - 2 p)) / sin(x) + 1)

- For p = 0,01, x ≈ 3,10, the minimal k ≈ 0,75.
- For p = 0,25, x ≈ 2,12, k ≈ 0.95.
- For p = 0,40, x ≈ 1,77, k ≈ 0,99.
- **In general, as p increases (0 < p < 0.5), the minimal k increases.**

Ant and Honey on the Cylinder

- Similar to the previous one, numerical analysis shows that **there is an optimal k also when p > 0.**
- We are looking for a **minimum of d / D**, i.e. a minimum of:

2 r sqrt(
 ((1 - 2 p) sin(x) + k sin(x) / (cos(x) + 1 / (1 - 2 p)))^2
 + ((1 - 2 p) cos(x) + k + 1)^2
)
/ (2 r (k + 1))

- From
 2 x - 2 k sin(x) / (cos(x) + 1 / (1 - 2 p)) - π = 0
 we get
 k = (x - π/2) (cos(x) + 1 / (1 - 2 p)) / sin(x)
 and include that in the above formula.

Ant and Honey on the Cylinder

- **So we are looking for a minimum of:**

sqrt(

((1 - 2 p) sin(x) + (x - π/2))^2

+

((1 - 2 p) cos(x) + (x - π/2)

(cos(x) + 1 / (1 - 2 p)) / sin(x) + 1)^2

)

/ ((x - π/2) (cos(x) + 1 / (1 - 2 p)) / sin(x) + 1)

- For p = 0,01, the minimum x ≈ 3,01,
  the optimal k ≈ 2,80 and d / D ≈ 0,89.
- For p = 0,25, x ≈ 2,77, k ≈ 3,56, d / D ≈ 0,95.
- For p = 0,40, x ≈ 2,44, k ≈ 5,65, d / D ≈ 0,99.
- **In general, as p increases (0 < p < 0.5),
  the optimal k increases and d / D approaches unity.**

In short –
The cylinder has only one shorter path, and

- **These regular prisms have at most**
  (depending on the parameters we denoted by k and p)
  this number of classes of shortest paths
  (paths that have the same length, so we count them as one):

- A **triangular prism** has up to 4 shortest paths for $2 \le k \le 5$.
  The length of one of them does not depend
  on the parameter p.
- A **tetrahedral prism** has up to 2 shortest paths for $2 \le k \le 5$.
- A **pentagonal prism** has up to 8 shortest paths for $2 \le k \le 5$.
  The length of one of them does not depend
  on the parameter p.
- A **hexagonal prism** has up to 4 shortest paths for $2 \le k \le 5$.

Some (my) hypotheses

- Note: when we say "shortest path", we actually mean the set of paths of the same length, shorter than the "obvious path".
- **Hypothesis H1**: Every odd regular prism has at least one p-independent shortest path, i.e. a shortest path whose length does not depend on the distance of points A and B (the ant and the honey) from the edge, while even prisms do not have one. The length is: **d = b sqrt(k^2 + 2 cos(π/n) k + 1)**
  We think we have a valid proof for this claim.
- **Hypothesis H2**: The number of shortest paths is unlimited as n increases and the parameter k remains limited, even when we allow only n rolls of the upper and lower base (i.e. we do not count spiral paths). We have a sketch of the proof.
- **Hypothesis H3**: The number of shortest paths is unlimited, for each n, when the parameter k increases indefinitely, but only if we allow an unlimited number of rolls of the upper and lower base (i.e. we allow spiral paths). We have a sketch.

ISTRA TECH (Croatia, Pula)
height h = 7 cm, base height b = 2.8 cm (k = 2.5), proportion of distance from edge p = 0.25 (0.7 cm); note: only the shortest paths are shown